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WAVEGUIDE EFFECT IN A ONE-DIMENSIONAL PERIODICALLY PENETRABLE STRUCTURE

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The waveguide properties of permeable one-dimensional periodic acoustic structures are studied here. These waveguide properties are associated with the existence of intrinsic waves localized in the vicinity of the structure. Their properties are described by generalized eigenfunctions which are solutions of problems describing the steady oscillations about the structure. The possibility of the existence of generalized eigenfunctions localized in the vicinity of a one-dimensional periodic penetrable layer or about a periodic chain of permeable barriers is demonstrated in this study. Examples are presented of the waveguide permeable periodic structures for which the boundaries asymptotic with respect to limited permeability or with respect to special geometric shape are studied, and also the properties of the natural oscillations, and eigenvalues are determined. These examples may serve as models both for experimental and numerical studies into the waveguide properties of a periodically permeable structure.

1. Formulation of the Problems and Necessary Information. Let a space be filled with a medium in which the speed of sound is represented by c_2 and the density in a state of rest is represented by ρ_2. The medium contains either a one-dimensional periodic layer (Fig. 1a) or a string of inclusions (Fig. 1b) of another medium, where the speed of sound is c_1, and the density in a state of rest is ρ_1. It is assumed that the boundary between these media is periodic along the y axis, with a period of 2π. It is assumed, further, that all motion within the media depends exclusively on two spatial variables: x, y. It is therefore convenient to utilize the following notation: Ω_1 is the area on the (x, y) plane which simulates the layer or chain of inclusions, while Ω_2 models the area filled with the external medium, and Γ represents the boundary between these media (see Fig. 1).

Let $f(x, y)\exp(-i\omega t)$ describe the periodic sources of the sound. It is assumed that the sources are situated in the medium Ω_2, positioned periodically along the y axis with a period 2π, ω is the angular frequency of the oscillations. The sound waves are described

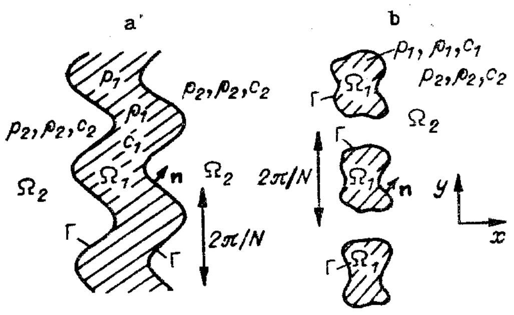

Fig. 1

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by acoustic perturbations in pressure $p_1(x, y, t)$ and $p_2(x, y, t)$ within the layer and in the external medium, while the steady-state acoustic oscillations are described by the functions $\mathrm{Re}[p_1(x, y)\exp(-i\omega t)]$ and $\mathrm{Re}[p_2(x, y)\exp(-i\omega t)]$, $p_1(x, y)$ and $p_2(x, y)$ must satisfy the Helmholtz equations in the corresponding regions and they must be complex-valued:

$$(\Delta + \lambda^2\varkappa^2)p_1 = 0 \text{ in } \Omega_1, \ (\Delta + \lambda^2)p_2 = f \text{ in } \Omega_2 \tag{1.1}$$

($\lambda = \omega/c_2$, $\kappa = c_2/c_1$, Δ is the Laplace operator). The sources are localized about the internal structure. This means that $f(x, y) \equiv 0$ for rather large values of the variable x.

At the boundary of contact between the media the conditions of continuity for pressure and for the gas-particle velocity component normal to the boundary must be satisfied, and **n** is the normal to Γ (see Fig. 1):

$$p_1 = p_2, \ \rho_2\partial p_1/\partial \mathbf{n} = \rho_1\partial p_2/\partial \mathbf{n} \text{ on } \Gamma. \tag{1.2}$$

Since c_1, ρ_1 and c_2, ρ_2 are constant in the corresponding areas, and the sources $f(x, y)$ and the boundary of contact between the media is 2π-periodic along the y axis, the acoustic fields described by the functions p_1 and p_2 will also be 2π-periodic along the y axis. The solution of Eqs. (1.1) must satisfy the radiation conditions [1-3]

$$p_2 = \sum_{k=+\infty}^{-\infty} a_k^{\pm} \exp\left(iky + i|x|\sqrt{\lambda^2 - k^2}\right), \quad |x| \gg 1 \tag{1.3}$$

(a_k^+ and a_k^- are certain complex numbers (and if $x > 0$, then we take a_k^+, while if $x < 0$, then a_k^-), k is an arbitrary whole number characterizing the number of the mode).

In the following, problem (1.1)-(1.3) for the determination of the acoustic field described by the functions p_1 and p_2, based on the familiar distribution of oscillation sources $f(x, y)$, will be referred to as the WG problem (waveguide). The WG problem represents a mathematical model describing the scattering of acoustic waves from periodic sources against a one-dimensional periodic structure. It is expedient to investigate the validity of this model, as well as the existence and singleness of the solutions.

The waveguide properties of the physical structure are defined by the eigenvalues and eigenfunctions of the WG problem. The properties of the mathematical model are studied with the aid of analytical operator-valued functions; such an approach allows us to employ the theory of the complex variable function. It must be noted that in the theory of diffraction at infinitely permeable structures the correct understanding of the mathematical essence of the problems is extremely important for the development of effective approximation methods as well. Following [1-3], we can assume that the function of the parameter λ, expressed through expression (1.3), is analytical on the infinite-sheeted Riemann surface Λ, i.e., its analytical continuation.

In the following discussion we will use

Definition 1.1. The quasieigenvalue of the WG problem is that element λ_* of the Riemann surface Λ for which the solution of this problem is not single-valued or that there exists a nontrivial solution of the uniform problem ($f \equiv 0$). The eigenvalue here is such a quasieigenvalue of λ_* for which the relationships $(\mathrm{Re}\,\lambda_*)\,(\mathrm{Re}\sqrt{\lambda_*^2 - k^2}) \geq 0$ are satisfied for all whole numbers k, given that $a_k^+ \neq 0$ or $a_k^- \neq 0$ under the radiation conditions from (1.3).

The concept of quasieigenvalues is auxiliary in nature, the physical sense of these values not clear to the end. The eigenvalues and eigenfunctions describe the waveguide property of the structure and are, generally speaking, generalized eigenfunctions, since the energy of these oscillations may be unbounded, if it is calculated over the entire oscillation region.

The qualitative behavior of the quasieigenvalues is described by the following assertion [1-4]:

THEOREM 1.1. The quasieigenvalues of the WG problem are discrete on the Riemann surface Λ. If the region Ω_2 is connected, the eigenvalues can only be real numbers.

LEMMA 1.1. The solution of the WG problem is a single one, provided that $\mathrm{Im}\,\lambda > 0$ and $\mathrm{Im}\sqrt{\lambda^2 - k^2} > 0$ for all k.

Proof. If we multiply similar ($f \equiv 0$) relationships (1.1) by the complex conjugate to p_1 and p_2 functions $\bar{p}_1$ and $\bar{p}_2$ and integrate over Ω_1 and Ω_2, then by means of the Green's theorem we can derive the relationships

$$\int_{\Omega_1} (\lambda^2 \varkappa^2 |p_1|^2 - |\nabla p_1|^2)\, d\Omega_1 + \int_{\Gamma} \bar{p}_1 (\partial p_1/\partial \mathbf{n})\, d\Gamma = 0,$$
$$\int_{\Omega_2} (\lambda^2 |p_2|^2 - |\nabla p_2|^2)\, d\Omega_2 - \int_{\Gamma} \bar{p}_2 (\partial p_2/\partial \mathbf{n})\, d\Gamma = 0.$$

Let $\tau = \rho_1/\rho_2$, $\tau > 0$, and $\sigma = \rho_2/\sigma_1$, $\sigma > 0$, and then, after multiplying the second of the relationships by τ (or the first relationship by σ) and combining the first equation with the second, taking into consideration the conditions of conjugacy (1.2), we can obtain the identity

$$I = \int_{\Omega_1} (\lambda^2 \varkappa^2 |p_1|^2 - |\nabla p_1|^2)\, d\Omega_1 + \tau \int_{\Omega_2} (\lambda^2 |p_2|^2 - |\nabla p_2|^2)\, d\Omega_2 \equiv 0.$$

Having equated the imaginary part to zero, we write

$$\operatorname{Im} I = (\operatorname{Im} \lambda^2) \left(\int_{\Omega_1} \varkappa^2 |p_1|^2\, d\Omega_1 + \tau \int_{\Omega_2} |p_2|^2\, d\Omega_2 \right) = 0. \tag{1.4}$$

Hence $p_1 \equiv 0$ and $p_2 \equiv 0$. The lemma is proved.

Remark 1.1. The physical sense of Lemma 1.1 lies in the fact that the nonexistence of free oscillations about a periodic structure in a medium with absorption has been proved. Relationships analogous to (1.4) can also be derived for the ratio of the densities σ. If the density of one of the media is equal to infinite in comparison with the other ($\tau = 0$, if $\rho_2 \to \infty$, or $\sigma = 0$, provided that $\rho_1 \to \infty$), the absence of free oscillations in a medium with absorption follows out of (1.4) and out of the conditions of conjugacy (1.2). All of the calculations have been carried out for a single period along y.

Let $\Phi(x, y, x_0, y_0, \lambda)$ represent the fundamental solution of the Helmholtz equation (1.1) in region Ω_2, satisfying the radiation conditions (1.3) and the conditions of periodicity along y with the period 2π [1-4]. We will subsequently use the following notation: $\psi = f * \Phi$, $*$ denotes convolution, $v = p - \psi$, $p = p_1$, if $(x, y) \in \Omega_1$ and $p = p_2$, if $(x, y) \in \Omega_2$. For the functions v in view of relationships (1.1) and (1.2) the following equations are satisfied:

$$(\Delta + \lambda^2) v_1 = \lambda^2 (1 - \varkappa^2)(v_1 + \psi) \text{ in } \Omega_1,\ (\Delta + \lambda^2) v_2 = 0 \text{ in } \Omega_2,$$
$$v_1 = v_2,\ \partial v_1/\partial \mathbf{n} = \tau \partial v_2/\partial \mathbf{n} + (\tau - 1)\partial \psi/\partial \mathbf{n} \text{ on } \Gamma \tag{1.5}$$

(v_1 and v_2 represent the narrowing of the v functions in regions Ω_1 and Ω_2). Since ψ and p satisfy radiation conditions (1.3), then v will also satisfy these conditions.

If we look for the solution of (1.5) in the form of the sum of the volumetric potential and the potential of a simple layer with densities $\rho(x, y)$ and $\nu(x, y)$, ρ is localized in Ω_1, ν is determined on Γ, the function $v(x, y)$ can be presented in the form

$$v(x, y) = \int_{\Gamma_0} \nu(x_0, y_0)\, \Phi(x, y, x_0, y_0)\, d\Gamma_0 + \int_{\Omega_1^0} \rho(x_0, y_0)\, \Phi(x, y, x_0, y_0)\, d\Omega_1^0. \tag{1.6}$$

Here and below, all the functions will be investigated in a single period such as, for example, in the band $\{0 \le y \le 2\pi\}$, $\Omega_j{}^0 = \{0 \le y \le 2\pi\} \cap \Omega_j$, $j = 1, 2$, $\Gamma_0 = \{0 \le y \le 2\pi\} \cap \Gamma$ (see Fig. 1).

In view of (1.6), the conditions of radiation for the functions $v(x, y)$ have been satisfied. The following equations, in light of (1.5), are valid for the unknown functions ρ and ν:

$$\nu = 2\left(\frac{\tau-1}{\tau+1}\right)\left[\int\limits_{\Gamma_0} \nu(\partial\Phi/\partial\mathbf{n})\,d\Gamma_0 + \int\limits_{\Omega_1^0} \rho(\partial\Phi/\partial\mathbf{n})\,d\Omega_1^0 + \partial\psi/\partial\mathbf{n}\right],$$
$$\rho = \lambda^2(1-\varkappa^2)\left[\int\limits_{\Gamma_0} \nu\Phi\,d\Gamma_0 + \int\limits_{\Omega_1^0} \rho\Phi\,d\Omega_1^0 + \psi(x,y)\right]. \tag{1.7}$$

We have [4, 5]

LEMMA 1.2. If ν and ρ satisfy (1.7), the soluton of the WG problem $p = v + \psi$, where v is defined from (1.6), and ψ represents the convolution of the fundamental solution of Φ with the function f, describing the sources of the acoustic oscillations. If $p(x, y)$ is the solution of WG problem, then $p = v + \psi$ and the functions ν and ρ corresponding to v and ψ satisfy relationship (1.7), which we will write as $(\nu, \rho) = T(\lambda, \tau, \kappa)\langle(\nu, \rho)\rangle(T(\lambda, \tau, \kappa)$, i.e., the linear operator corresponding to (1.7) in Hilbert space H, $H = L^2(\Gamma_c) \times L^2(\Gamma_0) \times L^2(\Omega_1{}^0)$).

With the aid of the data from [5], Lemma 1.1 and Lemma 1.2 have been proved.

LEMMA 1.3. The operator $T(\lambda, \tau, \kappa)$: $H \to H$, defined in (1.7), depends analytically on the parameter λ at the Riemann surface Λ and continuously depends on the real parameters τ and κ, $0 \le \tau < 1$, $\kappa^2 \ne 1$ in strong operator norm. There exists an element λ_* of the Riemann surface Λ for which the solution of (1.7) exists singly for all values of the parameters τ, $0 \le \tau < 1$, and κ, $\kappa^2 \ne 1$.

For continued discussion we require

THEOREM 1.2. If the ratio $\rho_1/\rho_2 = \tau$ tends to zero ($\tau \to 0$), then for sufficiently small τ the quasieigenvalues of $\lambda_*(\tau)$ of the WG problem exist and converge in the topology of the Riemann surface Λ either toward ν_* if $\nu_*{}^2\kappa^2$, i.e., the eigenvalue of the Laplace operator in region Ω_1 for functions satisfying the conditions of 2π-periodicity along y and the uniform Neumann conditions at the boundaries Γ of this region $\left(\lim\limits_{\tau\to 0}\lambda_*(\tau) = \nu_*\right)$ or to the quasieigenvalues μ_{ext} of the Dirichlet problem for the Helmholtz equations in the region Ω_2 $\left(\lim\limits_{\tau\to 0}\lambda_*(\tau) = \mu_{ext}\right)$.

If the ratio $\rho_2/\rho_1 = \sigma$ tends toward zero ($\sigma \to 0$), for sufficiently small σ the quasieigenvalues of $\lambda_*(\sigma)$ exist in the WG problem and converge on μ_* $\left(\lim\limits_{\sigma\to 0}\lambda_*(\sigma) = \mu_*\right)$ if $\mu_*{}^2\kappa^2$ are the eigenvalues of the Laplace operator in the region Ω_1 with the conditions of Dirichlet on Γ and with the conditions of 2π-periodicity for the eigenfunctions, or to the quasieigenvalues of ν_{ext} in the Neumann problem in the region Ω_2 $\left(\lim\limits_{\sigma\to 0}\lambda_*(\sigma) = \nu_{ext}\right)$.

Proof. In view of Lemma 1.3 the conditions of Theorem 7.2 [5, p. 381] have been satisfied. It can therefore be assumed that the quasieigenvalues of λ_* in the WG problem depend continuously on the parameter τ, $0 \le \tau < 1$. When $\tau = 0$, relationships (1.7) are equivalent to a problem of the form

$$(\Delta + \lambda^2\varkappa^2)p_1 = 0 \text{ in } \Omega_1,\ (\Delta + \lambda^2)p_2 = f \text{ in } \Omega_2,$$
$$p_1 = p_2,\ \partial p_1/\partial\mathbf{n} = 0 \text{ on } \Gamma. \tag{1.8}$$

The functions f, p_1, and p_2 satisfy the periodicity conditions, and p_2, moreover, satisfies the radiation conditions. Since relationships (1.8) for the p_1 function represent the Neumann problem in region Ω_1 with conditions of 2π-periodicity along y for the solutions, and in the case of the p_2 function, representing the Dirichlet problem for the Helmholtz equations in Ω_2 with the radiation conditions at infinity, then with τ close to zero, owing to continuity from τ, the quasieigenvalues corresponding to the WG problem are close either to ν_* or to μ_{ext}. The second part of the theorem regarding the continuous dependence on parameter σ, $0 \le \sigma < 1$, with respect to the quasieigenvalues of the WG problem is validated on the basis of analogous considerations. The theorem is proven.

Let the investigated permeable structure Ω_1 with boundary Γ have a period $2\pi/N$, N is a natural number greater than unity. In this case, if $p_*(x, y)$ is a quasieigenfunction of the WG problem, $p_*(x, y + 2\pi n/N)$ will also be a quasieigenfunction of the WG problem for all whole numbers n and N. Let u(y) be a 2π-periodic function along y, whose expansion into a Fourier series for any natural number N can be written as follows:

$$u(y)=\sum_{k=-\infty}^{+\infty} a_k \exp(iky)=\sum_{n=1}^{N}\sum_{k=-\infty}^{+\infty} a_{n+kN}\exp[iy(n+kN)]=\sum_{n=1}^{N} u_n(y).$$

By definition the function $u_n(y)$ satisfy the conditions of quasiperiodicity

$$u_n(y+2\pi/N)=u_n(y)\exp(i2\pi n/N). \tag{1.9}$$

Let us note that these relationships describe the specifics of oscillation about a $2\pi/N$-periodic structure. We generally say that (1.9) describes a shift in the phase of the oscillations to adjacent regions of the structure.

Let $p_n(x, y)$ be the solution of the WG problem for which the conditions of quasiperiodicity in (1.9) are valid, and in this event the radiation conditions have the form

$$p_n(x, y, \lambda)=\sum_{k=-\infty}^{+\infty} c_k^{\pm}\exp\left[i(n+kN)y+i|x|\sqrt{\lambda^2-(n+kN)^2}\right]. \tag{1.10}$$

Functions of λ such as (1.10) are analytical on the Riemann surface Λ_n of their analytical extension. The branching point of Λ_n will be the numbers $\pm(n + kN)$ for all whole k. Let $\Lambda_n{}^0$ be such a sheet of the Riemann surface Λ_n with sections $(-\infty, -\min\{n, N-n\}]$ and $[\min\{n, N-n\}, +\infty)$, where the inequalities for all whole k are satisfied: $\mathrm{Im}\sqrt{\lambda^2-(n+kN)^2} > 0$. From the standpoint of application the following is essential to the discussion.

THEOREM 1.3. If the quasieigenfunction $p^*(x, y, \lambda_*)$ of the WG problem satisfies the conditions of quasiperiodicity in (1.9) and the corresponding quasieigenvalue of λ_* is found on the shape $\Lambda_n{}^0$ of the Riemann surface Λ_n, then $p^*(x, y)$ diminishes with increasing distance from the barrier and is an eigenfunction, while λ_* is a real eigenvalue of the WG problem.

Proof. Owing to the determination of the sheet $\Lambda_n{}^0$ all of the terms in the expression (1.10) for the function $p^*(x, y)$ diminish with increasing distance from the barrier Ω_1. From relationships such as (1.4) it follows for $p^*(x, y)$ that λ_* is a real number. Since $\lambda_* \in \Lambda_n{}^0$, then $|\lambda_*| < \min\{n, N-n\}$, which means that $\mathrm{Re}\sqrt{\lambda_*{}^2-(n+kN)^2} = 0$ for all whole k. QED

2. Waveguide Effect of a Permeable Layer. Let the region Ω_1 simulate the layer being penetrated (see Fig. 1a). It is assumed that Ω_1 is periodic along the y axis with a period $2\pi/N$. For various applied problems it is expedient to investigate the quasiperiodic solutions of the WG problem for certain numbers n and N, $1 \le n < N$, from condition (1.9). Let the velocity c_1 of wave propagation in the medium filling the Ω_1 layer be less than the velocity c_2 of wave propagation in the medium Ω_2, $\kappa = c_2/c_1 > 1$. In this case the following is valid.

THEOREM 2.1. For sufficiently large κ and sufficiently small τ (or σ) the Ω_1 layer exhibits the waveguide effect.

Proof. Let $\nu_k{}^2\kappa^2$ be the eigenvalues of the Neumann problem for the Helmholtz equation in the Ω_1 region as a class of functions satisfying the conditions of quasiperiodicity (1.9) for certain n and N and $|\nu_1| \le |\nu_2| \le \ldots$ In view of Theorem 1.2 for each fixed k there exist quasieigenvalues of λ_k in the WG problem with quasiperiodicity conditions such that there exist a limit transition in the topology of the Riemann surface Λ_n, $\nu_k=\lim_{\tau\to 0}\lambda_k(\tau)$. Since κ may be a rather large number, there exist such ν_k (at least one number) for which the inequality $|\nu_k| < \min\{n, N-n\}$, $k = 1, \ldots, k_0$ is valid. Quasieigenvalues of λ_k for the WG problem corresponding to ν_k near $|\lambda_k| < \min\{n, N-n\}$, $k = 1, \ldots, k_0$ means that for sufficiently small τ the inequality λ_k, $k = 1, \ldots, k_0$ is valid. It may therefore be assumed that for sufficiently small τ the quasieigenvalues of λ_k, $k = 1, \ldots, k_0$ belong to the sheet $\Lambda_n{}^0$ of the Riemann surface Λ_n. It follows from Theorem 1.3 that λ_k are real eigenvalues of the WG problem, and the eigenfunctions corresponding to λ_k are localized in the vicinity of the penetrated layer Ω_1, $k = 1, \ldots, k_0$. Let $\mu_k{}^2\kappa^2$ be the eigenvalues of the Dirichlet problem for the Laplace operator in the Ω_1 region as a class of functions satisfying the conditions of quasiperiodicity (1.9) for certain fixed N and n; k_0, n, and N are natural numbers. In view of Theorem 1.2 there exist quasieigenvalues of λ_k in the WG problem with quasiperiodicity conditions such that $\lim_{\sigma\to 0}\lambda_k(\sigma)=\mu_k$ in the topology of the

Riemann surface Λ_n. For any fixed geometry of Ω_1 and fixed n and N there exists a sufficiently large number κ such that for certain k, $k = 1, \ldots, k_0$ the inequality $|\mu_k| < \min\{n, N - n\}$ is satisfied. Therefore, for sufficiently small σ the quasieigenvalues of λ_k, $k = 1, \ldots, k_0$, corresponding to μ_k, belong to the sheet $\Lambda_n{}^0$ of the Riemann surface Λ_n. In view of Theorem 1.3, these quasieigenvalues are both eigenvalues and real. The theorem has been proved.

Remark 2.1. Here and below, according to R. M. Garipov, the waveguide effect of a one-dimensional periodically penetratable structure will be understood to refer to the existence of generalized eigenfunctions localized in the vicinity of that structure.

Remark 2.2. If the quasieigenfunction of the WG problem is localized in the vicinity of the structure Ω_1, then the corresponding quasieigenvalues are eigenvalues and real numbers.

Let the layer Ω_1 have the shape shown in Fig. 2 and let it satisfy conditions of $2\pi/N$-periodicity along the y axis, c_1 and c_2 are fixed, and ε is the transverse dimension of the constriction. In this event the following is valid.

THEOREM 2.2. The Ω_1 layer described in Fig. 2, for sufficiently small τ and ε, exhibit the waveguide effect.

Proof. Let $\xi_k = \nu_k{}^2\kappa^2$ be the eigenvalue of the Neumann problem for the Laplace operator in the region Ω_1 in the class of functions satisfying the quasiperiodicity conditions (1.9). For sufficiently small ε there exist [5] ξ_k which are close to the eigenvalues of the Neumann problem for the Laplace equation in the finite subregions of Ω_1, derived from Ω_1 for $\varepsilon = 0$. Since zero is the eigenvalue of the Neumann problem, for sufficiently small ε there exist always at least one (or several) ξ_k, $k = 1, \ldots, k_0$, such that $|\nu_k| < \min\{n, N - n\}$. On the strength of Theorem 1.2 there exist quasieigenvalues of λ_k in the WG problem such that $\nu_k = \lim\limits_{\tau\to 0} \lambda_k(\tau)$, and for sufficiently small τ we can therefore assume that $\lambda_k(\tau)$, $k = 1, \ldots, k_0$ are situated on the $\Lambda_n{}^0$ sheet of the Riemann surface Λ_n. Consequently, the conditions of Theorem 1.3 have been satisfied, which was what had to be proved.

Theorems 2.1 and 2.2 are connected to the "internal" geometry of the region Ω_1, which is "taken into consideration" by the eigenvalues of the Dirichlet problem or by the Neumann problem for the Laplace operator in the region Ω_1. The theorem which follows below is connected to the "external" properties of the region Ω_1.

Let the Ω_1 layer exhibit a geometry such as that shown in Fig. 3a-c, the period of Ω_1 along the y axis being equal to $2\pi/N$, c_1 and c_2 are certain arbitrary fixed numbers and

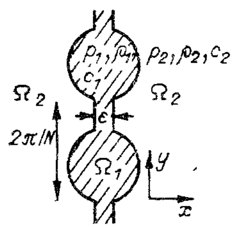

Fig. 2

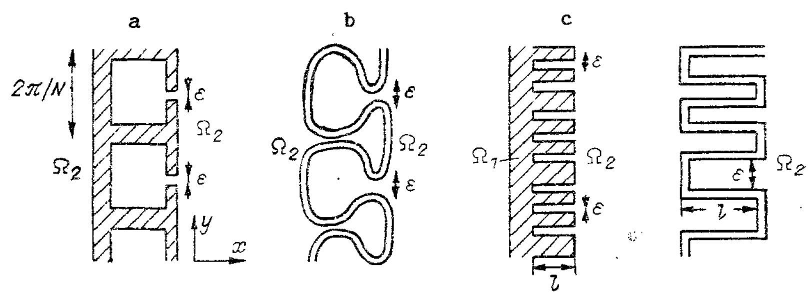

Fig. 3

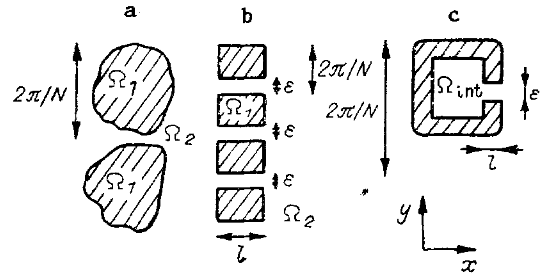

Fig. 4

ε represents the width of the resonator neck in cases a and b of Fig. 3 or the transverse dimension of the half-open channel (Fig. 3c). We then have

THEOREM 2.3. The Ω_1 layer in Fig. 3 for sufficiently small σ and ε exhibits the waveguide effect.

Proof. On the strength of Theorem 1.2 for sufficiently small σ the quasieigenvalues of λ_k in the WG problem are close to the quasieigenvalues of ν_k in the Neumann problem for the region Ω_2. Given sufficiently small ε the quantities ν_k are close in the topology of the Riemann surface Λ_n to the corresponding numbers describing the intrinsic oscillations in the resonator ($\varepsilon = 0$), or in the neck of the resonator, or in the half-open channel [2]. For arbitrary n and N, $1 \le n < N$, for sufficiently small ε and σ, $\ell \gg 1$ the inequality $|\lambda_k| < \min\{n, N - n\}$, $k = 1, \ldots, k_0$ is valid. It might be assumed that for small ε and σ all λ_k, $k = 1, \ldots, k_0$ are found on the $\Lambda_n{}^0$ sheet of the Riemann surface Λ_n. The conditions of Theorem 1.3 for the quasieigenvalues of λ_k, $k = 1, \ldots, k_0$ in the WG problem are satisfied and Theorem 2.3 is proved.

3. Waveguide Effect for a Discontinuous Periodic Structure. Let Ω_2 be a connected region. In this case Ω_1 has several connected components in the 2π-period and describes the periodic connection of medium 1 to medium 2 (Fig. 4a-c). We have used the following notation: Ω_ε, the channel or orifice; Ω_{int}, the interior of the resonators ($\varepsilon = 0$) (Fig. 4c); Ω_{ext}, the exterior of the resonators; $\Omega_2 = \Omega_{int} + \Omega_\varepsilon + \Omega_{ext}$. Figure 4a shows that $\Omega_{int} = \phi$, $\Omega_\varepsilon = \phi$, $\Omega_{ext} = \Omega_2$; Fig. 4b shows that $\Omega_{int} = \phi$, $\Omega_2 = \Omega_\varepsilon + \Omega_{ext}$.

Let the region Ω_1 consist of N connected components in the band $0 \le y \le 2\pi$ and, moreover, it is assumed to be periodic along the y axis with a period $2\pi/N$. We have

THEOREM 3.1. The discontinuous one-dimensional periodic structure of Ω_1 for sufficiently small τ exhibits the waveguide effect.

Proof. On the strength of Theorem 1.2 there exist quasieigenvalues of λ_k in the WG problem with conditions (1.9) such that $\lim\limits_{\tau\to 0} \lambda_k(\tau) = \nu_k$, provided that $\xi_k = \nu_k{}^2\kappa^2$ is the eigenvalue of the Laplace operator in one of the connected components of region Ω_1 with the Neumann conditions at the boundaries of this region. Let ξ_k, $k = 1, \ldots, k_0$ be such eigenvalues for which the inequality $|\nu_k| < \min\{n, N - n\}$, $k = 1, \ldots, k_0$ is valid. Then, for sufficiently small τ there exists the inequality $|\lambda_k| < \min\{n, N - n\}$, $k = 1, \ldots, k_0$ for some number n, $1 \le n < N$. Therefore, for the numbers λ_k the conditions of Theorem 1.3 have been satisfied, which is what had to be proved.

Remark 3.1. Since zero is the eigenvalue of the Laplace operator with the Neumann conditions at the boundaries of Ω_1, for any κ there exist at least one number λ_k such that $|\lambda_k| < \min\{n, N - n\}$ for all sufficiently small τ.

Let a chain of penetrable barriers have the form shown in Fig. 4b, c. The quantities ε and ℓ characterize the width and length of the Ω_ε channels. The following is then valid.

THEOREM 3.2. The one-dimensional periodic chain of penetrable barriers Ω_1 (see Fig. 4b, c) for large ℓ and sufficiently small σ and ε exhibits the waveguide effect.

Proof. On the strength of Theorem 1.2 for small σ the quasieigenvalues of λ_k in the WG problem are close to the quasieigenvalues of ξ_k for the Neumann problem with the Helmholtz equation in region Ω_2. The quasieigenvalues of ξ_k for the Neumann problem in Ω_2 on the basis of [2] for sufficiently small ε are close to the numbers ν_k, if $\nu_k{}^2$ represents the eigenvalues of the Neumann problem for the Laplace operator in the Ω_{int} region (see Fig. 4c) or $\nu_k = k\pi/\ell$ (Fig. 4b, c). In the latter case ν_k describe the intrinsic oscillations in the Ω_ε channel. Let ν_k, $k = 1, \ldots, k_0$ be numbers such that $|\nu_k| < \min\{n, N - n\}$, in which case, for sufficiently small ε and σ, the quasieigenvalues of $\lambda_k(\varepsilon, \sigma)$ in the WG problem satisfy the inequality $|\lambda_k| < \min\{n, N - n\}$, $k = 1, \ldots, k_0$. And in this case as well, for the WG problem, the conditions of quasiperiodicity (1.9) are valid, the conditions of Theorem 1.3 having been satisfied. It may therefore be assumed that λ_k belongs to the Λ_n^0 sheet of the Riemann surface Λ_n, which is what had to be proved.

The study of the waveguide properties of various structures has important applied significance. However, the author knows of no studies which have investigated the waveguide properties of one-dimensional periodically penetrable structures. Investigations into the waveguide properties of structures that are uniform with respect to one of the variable parameters can be found in [6]. These investigations were further developed in [7, 8].

4. Example. Waveguide Property of Cylindrical Air Cavities in Aluminum. Let the structure in Fig. 4b describe cylindrical air cavities in aluminum: $c_1 = 330$ m/sec, $c_2 = 5200$ m/sec, and $\kappa = 15.7$, with densities of $\rho_1 = 1.21$ kg/m^3, $\rho_2 = 2700$ kg/m^3, with $\tau = 0.00045$. We note that the values of the parameters satisfy the conditions of Theorem 3.1. Therefore, in approximate terms we can calculate the eigenvalues of this structure. Let all of the cylindrical cavities be periodic along the y axis and let them have the identical radius R, the distances between the centers of the cavities represented by H (H > 2R and the cavities are not in contact with each other). If on a scale of a single wavelength along the y axis there are N cavities, the wavelength L = NH. The conditions of radiation (1.3) are written as follows:

$$p_2 = \sum_{k=-\infty}^{+\infty} a_k^{\pm} \exp\left[(iy2k\pi/NH) + i|x|\sqrt{\lambda^2 - (2\pi k/NH)^2}\right] \tag{4.1}$$

($\lambda = \omega/5200$). The eigenvalues for ξ_k of the Laplace operator in a circle of radius R with the Neumann conditions at the boundary can be found from tables presented in [9]. Then $\xi_0 = 0$; $\xi_1 R = 3.832$; $\xi_2 R = 7.016$; $\xi_3 R = 10.173$; $\xi_1 = R^{-1}\cdot 3.832$; $\xi_2 = R^{-1}\cdot 7.016$; $\xi_3 = R^{-1}\cdot 10.173$. With the aid of the relationship $\xi_k = \nu_k^2\kappa^2$, presented in the proof of Theorem 3.1, we can calculate $\nu_k = \sqrt{\xi_k\kappa^{-2}}$. Consequently, $\nu_0 = 0$; $\nu_1 = 0.1247/\sqrt{R}$; $\nu_2 = 0.169/\sqrt{R}$; $\nu_3 = 0.203/\sqrt{R}$. Since on the strength of Theorem 3.1, $\lambda_k(\tau) \to \nu_k$, as $\tau \to 0$ and $\lambda = \omega/c_2$, we will calculate the circular frequency $\omega_k = \lambda_k c_2$; $\omega_0 = 0$; $\omega_1 = 648.4/\sqrt{R}$; $\omega_2 = 878.8/\sqrt{R}$; $\omega_3 = 1055.6/\sqrt{R}$. If R = 0.005 m, the resonance frequencies are equal to $\omega_1 = 9169.7$, $\omega_2 = 12{,}428$; $\omega_3 = 14{,}928$. For convenience in calculation and experimental verifications, let the distance between the centers of the orifices be H = 0.02 m. Let us also assume that the oscillations in the adjacent orifices are in counterphase, and in this case we can assume in (1.9) that N = 2, n = 1. According to the proof of Theorem 3.1, the eigenvalues must satisfy the inequality which follows from (4.1):

$$|\lambda_k| < \min\{n\pi/H,\ N\pi/H - n\pi/H\} = 157.0,$$
$$\nu_1 = 0.1247/\sqrt{0.0005} = 1.763,\ \nu_2 = 2.39,\ \nu_3 = 2.87.$$

Since $\lambda_k \to \nu_k$ as $\tau \to 0$, we note that all of the found frequencies may govern the possible waveguide property of the structure. For the sake of experimental convenience we can calculate, in approximate terms, the velocity of propagation for the intrinsic modes along the y axis. In these cases the wave number is equal to π/H, while the propagation velocity v_k for the corresponding intrinsic mode is $v_k = \omega_k H/\pi$ (or $v_1 = 58.376$ m/sec, $v_2 = 79.12$ m/sec, $v_3 = 95.03$ m/sec). We know of no experimental studies into the decelerating and waveguide properties of periodic acoustic structures.

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SELF-OSCILLATION REGIMES IN A SYSTEM OF FOUR QUASI-TWO-DIMENSIONAL VORTICES

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Particular attention has recently been devoted to experimental studies of transitional processes in the appearance of turbulence in simple hydrodynamic flows. In the present study we present a model of the elementary cell of a quasi-two-dimensional double-period flow that is related to Kolmogorov flow [1-4]. The derived results may prove to be useful, for example, in application to the problem of constructing limited-mode systems which, in basic outline, describe the nonlinear processes occurring in hydrodynamic flows [2, 3].

The primary flow regime is a steady system of four quasi-two-dimensional vortices. The self-oscillations in such a system were first detected in studies of the convective motion in a Haley—Shaw cell [5, 6], and subsequently in a uniform fluid in which the flow was induced by means of a magnetohydrodynamic drive [7-9].

It is the aim of the present study to further investigate the above-indicated system of vortices. The flow is generated in a horizontal rectangular cuvette in layers of various thicknesses, under the action of an MHD force periodic along both coordinates. In particular, we have derived the relationship between the amplitudes of the self-oscillations and the Reynolds number, and a spectral analysis of the self-oscillation regimes has been carried out. We have examined the effect of friction against the bottom on the characteristics of the flow.

1. Laboratory Equipment and the Experimental Method. The experiments were conducted on an installation such as that described in [9]. The flow was established within a rectangular cuvette having dimensions of 24 × 12 × 3 cm, positioned horizontally on a Plexiglas frame. Two three-pole electromagnets are contained symmetrically within the frame. The magnetic field induction **B** of the electromagnets within the region of the cuvette has a vertical component which can approximately be presented in the form

$$B_z(x,\ y,\ z) = B_0(z)\sin(2\pi x/L_x)\cos(2\pi y/L_y).$$

Here $B_0(z)$ is the amplitude $B_z(x, y, z)$ on the plane z = const; L_x = 24 cm and L_y = 12 cm represent the length and width of the cuvette along the x and y axes, lying on the plane z = 0 and coincident with the two adjacent sides of the cuvette. The z axis is directed vertically upward. An electrolyte (a $CuSO_4$ solution with a concentration of 100 g/liter)

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